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Extended Class of Metric Tensors in Relativity

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An extended "metric" tensor that is a function of an internal vector $y^a(x)$ leads to a spin-1 massless field of gravitational origin. It is shown that this new field vanishes in the linear approximation for the extended "metric."

1. INTRODUCTION

In general relativity, that is, in a four-dimensional Riemannian space-time, the metric is a symmetric second-rank tensor which is a function of the coordinates x^μ and is a solution of Einstein's field equations. Associated to the metric there exists a field of local vierbeins $e^a_\mu(x)$, such that locally the metric becomes Lorentzian. In this paper we use the following notation: Latin indices indicate internal, or tetrad indices, and Greek indices indicate space-time degrees of freedom. Both vary from 0 to 3.

Here we propose a generalization of this geometrical structure such that the "metric" becomes a function of an internal vector field $y^a(x)$. By internal vector we mean an aggregate of four real space-time functions, in a Riemannian space-time, which transform under the action of local $SO(3, 1)$ transformations similarly to the field of local vierbeins. It should be mentioned that extended "metrics" of the form $g_{\mu\nu}(x^\alpha, y^\alpha(x))$ have been considered in Finsler geometry (Cartan, 1934; Horvath and Gyulai, 1956; Matsumoto, 1971; Ikeda, 1979). However, here we consider the y's as the components of an internal vector, that means, they depend on the y^α through the combination $y^a = y^\alpha e^a_\alpha$. Besides this, the results derived in this paper are different from those valid for the Finsler geometry.

In general, we consider geometrical objects of the form $T^{a\cdots}_{\mu\cdots}(x, y)$, where $y = (y^a(x))$, such that the "metric" $g_{\mu\nu}(x, y)$ becomes a particular case of this general set of objects.

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In Section 2 we define the full covariant differential $\underset{(f)}{\Delta} g_{\mu\nu}(x, y)$ of the "metric," and determine the metrical conditions. The definition of these metrical conditions implies the existence of a space-time vector $\phi_\rho(x)$. It may be shown that associated with this new vector field there exists an internal transformation of the form $\bar{y} = \lambda y$ such that the variation $\delta\phi_\rho$ takes the form

$$\delta\phi_\rho = \frac{\partial}{\partial x p} \ln \lambda$$

in analogy with the gauge transformation of an electromagnetic potential. In the Section 3 the curvatures associated to the several covariant derivatives are determined. One of these curvatures has the form $F_{\rho\lambda} = 2\partial_{[\lambda}\phi_{\rho]}$, such that $\delta F_{\rho\lambda} = 0$ under the previous dilatation transformation of the y^a. Thus, the $F_{\rho\lambda}$ may be interpreted as the field strength associated to ϕ_ρ. In Section 4 the properties of this new vector field are discussed, and a Lagrangian density for the system $g_{\mu\nu}(x)$ and $\phi_\rho(x)$ is suggested. Finally, in Section 5 we consider the weak field approximation for the potentials $g_{\mu\nu}(x)$ and $\phi_\rho(x)$.

2. THE METRICAL CONDITIONS

Consider the class of functions $T^{a\dots}_{\mu\dots}(x, y)$. The differential of such quantities is given by

$$dT^{a\dots}_{\mu\dots} = \left[\left(\frac{\partial T^{a\dots}_{\mu\dots}}{\partial x^\lambda}\right)_{\exp} + \frac{\partial T^{a\dots}_{\mu\dots}}{\partial y^b}\frac{\partial y^b}{\partial x^\lambda}\right] dx^\lambda$$

(The symbol "exp" denotes the explicit derivatives with respect to the coordinates x^ρ.)

In this expression only the term containing the derivatives of $T^{a\dots}_{\mu\dots}$ with respect to y^b transforms as a collection of internal tensors. The remaining terms need to be corrected in order to obtain a covariant expression under $SO(3, 1)$ transformations. With this in mind, we introduce a full covariant differential under internal transformations, according to the expression

$$\underset{(i)}{\Delta} T^{a\dots}_{\mu\dots} = \left[T^{a\dots}_{\mu\dots|\rho} + \frac{\partial T^{a\dots}_{\mu\dots}}{\partial y^b} y^b_{\|\rho}\right] dx^\rho \tag{1}$$

We recall that local $SO(3, 1)$ transformations are those associated with pseudo-orthogonal internal matrices $L = (L^a_b(x))$. The two types of covariant derivatives in equation (1) are defined by

$$T^{a\ldots}_{\mu\ldots|\rho}=\left(\frac{\partial T^{a\ldots}_{\mu\ldots}}{\partial x^{\rho}}\right)_{\exp}+\Gamma^{a}_{.b\rho}T^{b\ldots}_{\mu\ldots} \tag{2}$$

$$T^{a\ldots}_{\mu\ldots\|\rho}=\frac{\partial T^{a\ldots}_{\mu\ldots}}{\partial y^{b}}\,y^{b}_{\|\rho},\qquad y^{b}_{\|\rho}=\frac{\partial y^{b}}{\partial x^{\rho}}+F^{b}_{.c\rho}y^{c} \tag{3}$$

The connection Γ_ρ is associated with the local $SO(3,1)$ transformations. Here this connection appears as the correction factor corresponding to the derivatives $(\partial_\rho T^{a\ldots}_{\mu\ldots})_{\exp}$. Using an analogy with the Finsler geometry we associate with the term $\partial_\rho y^b$ another connection denoted by F_ρ. However, since this choice is merely an analogy, and since both Γ_ρ and F_ρ transform similarly under local $SO(3,1)$ transformations,[3] we necessarily have that the difference $F_\rho-\Gamma_\rho$ is an internal tensor. This tensor will be denoted by τ_ρ. Thus, expression (1) gives the definition of a covariant internal derivative for the class of functions $T^{a\ldots}_{\mu\ldots}(x,y)$ in a Riemannian space-time. [By the term *Riemannian* we mean the space with metric $g_{\mu\nu}(x)$; it should be noted that here the meaning of the term *Riemannian* has a weaker sense since the metrical conditions are not yet determined.]

Following an analogous process one may introduce a full covariant differential by replacing $T^{a\ldots}_{\mu\ldots|\rho}$ by $T^{a\ldots}_{\mu\ldots;\rho}$ and $\underset{(i)}{\Delta}$ by $\underset{(f)}{\Delta}$,[4] where

$$T^{a\ldots}_{\mu\ldots;\rho}=T^{a\ldots}_{\mu\ldots|\rho}-\Omega^{\lambda}_{.\mu\rho}T^{a\ldots}_{\lambda\ldots}$$

and assuming that Γ_ρ and F_ρ are a set of world vectors. The quantities $\Omega^{\lambda}_{.\mu\rho}$ indicate the components of the space-time connection. The explicit value for Ω will be derived from the metricity conditions.

We use the operator notation

$$\underset{(f)}{\Delta}=dx^{\rho}D_{\rho}$$

For instance, for the extended "metric" $g_{\mu\nu}(x,y)$ we have

$$D_\rho g_{\mu\nu}(x,y)=g_{\mu\nu;\rho}+\frac{\partial g_{\mu\nu}}{\partial y^{a}}\tau^{a}_{.b\rho}y^{b}+\frac{\partial g_{\mu\nu}}{\partial y^{a}}y^{a}_{|\rho} \tag{4}$$

[3]This is necessary since we have assumed that $\underset{(i)}{\Delta}T^{a\ldots}_{\mu\ldots}(x,y)$ transforms as an internal vector under local Lorentz transformations, viz.

$$\underset{(i)}{\Delta}T'^{a\ldots}_{\mu\ldots}(x,y')=L^{a}_{.b}(x)\underset{(i)}{\Delta}T^{b\ldots}_{\mu\ldots}(x,y)$$

$$y'=L.y$$

[4]We remind the reader that under coordinate transformations the $y^a(x)$ behave as a set of scalars. Therefore in the definition of $y^{b}_{\|\rho}$ both factors on the right-hand side are world vectors, assuming as usual that F_ρ is a collection of world vectors.

Presently, we will take $g_{\mu\nu}(x, y)$ of the form

$$g_{\mu\nu}(x, y) = e^a_\mu(x) e^b_\nu(x) \omega_{ab}(y) \tag{5}$$

The $e^a_\mu(x)$ indicate the field of vierbeins. The expression (5) differs from the conventional formula giving the metric $g_{\mu\nu}(x)$ only by the factor $\omega_{ab}(y)$ which substitutes the local Lorentzian metric η_{ab}.

In what follows we will postulate the following values for ω and τ:

$$\omega_{ab}(y) = \eta_{ab}\psi(y) \tag{6}$$

$$\tau^a_{.b\rho} = -\delta^a_b \phi_\rho \tag{7}$$

From equations (5) and (6) we get

$$g_{\mu\nu}(x, y) = \psi(y) g_{\mu\nu}(x) \tag{8}$$

Thus, $g_{\mu\nu}(x, y)$ may be looked as the result of an active "conformal transformation" on the metric $g_{\mu\nu}(x)$, induced by the field $y(x)$. We may say that the space with "metric" $g_{\mu\nu}(x, y)$ is "conformally Riemannian." Replacing expression (8) in equations (4) we get

$$D_\rho g_{\mu\nu}(x, y) = \psi(y) g_{\mu\nu;\rho}(x) + g_{\mu\nu}(x)\left(\frac{\partial\psi}{\partial y^a} y^a_{|\rho} - \frac{\partial\psi}{\partial y^a} y^a \phi_\rho\right) \tag{9}$$

In these equations we have made the supposition that $\Omega(x, y) = \Omega(x)$. Consequently $g_{\mu\nu;\rho}(x)$ corresponds to the covariant derivative associated with space-time coordinate transformations. We mentioned that these conditions on the connection Ω are valid for the Christoffel symbols associated with the explicit dependence on the coordinates x^λ of the metric $g_{\mu\nu}(x, y)$ given by (8).

We now impose that $\psi(y)$ is homogeneous of the first degree in the variable y^a:

$$y^a \frac{\partial\psi}{\partial y^a} = \psi \tag{10}$$

Accordingly, equations (9) may be written as

$$D_\rho g_{\mu\nu}(x, y) = \psi(y)\left\{ g_{\mu\nu;\rho}(x) + g_{\mu\nu}(x)\left[\frac{\partial(\ln\psi)}{\partial y^a} y^a_{|\rho} - \phi_\rho\right]\right\} \tag{11}$$

In the definition of the metricity conditions we may use two possibilities:

(a) The derivatives $D_\rho g_{\mu\nu}(x, y)$ determine a Weyl field k_ρ according to

$$D_\rho g_{\mu\nu}(x, y) = k_\rho g_{\mu\nu}(x, y) \tag{12}$$

Then,

$$g_{\mu\nu;\rho}(x) = (k_\rho - \Lambda_\rho) g_{\mu\nu}(x) \tag{13}$$

where

$$\Lambda_\rho = \frac{\partial(\ln \psi)}{\partial y^a} y^a_{|\rho} - \phi_\rho$$

However, such Weyl field in the equation (13) depends on the quantities k_ρ and ϕ_ρ which are not related to each other. Thus, such interpretation contains two vector fields and a scalar field which are not determined.

(b) Another possible choice, which avoids the presence of undetermined quantities, is to take the space-time affinity as the Christoffel symbols associated to the explicit dependence of $g_{\mu\nu}(x, y)$ in the coordinates x^λ. This choice is consistent with equations (9). Along with such definition we also take $k_\rho = 0$. Accordingly, from equation (13) one gets $\Lambda_\rho = 0$, which implies that

$$\phi_\rho = \frac{\partial(\ln \psi)}{\partial y^a} y^a_{|\rho} = \phi_\rho(x, y) \tag{14}$$

Thus, the field ϕ_ρ becomes a function of the vector y^a which belongs to the internal space associated to a Riemannian space-time since the conditions $g_{\mu\nu;\rho}(x) = 0$ hold. In what follows we shall use the equation (14) for the definition of the field ϕ_ρ. From equations (8) and (10) it follows that $g_{\mu\nu}(x, y)$ is homogeneous of the first degree in the variables y^a. This result implies that the present approach is different from the method used in the Finsler geometry.

Consider the transformation, in a fixed space-time point and for a fixed Lorentz frame

$$\tilde{\psi}(\tilde{y}) = \lambda(y)\psi(y), \qquad y^a \frac{\partial \lambda}{\partial y^a} = 0 \tag{15}$$

$$\tilde{y}^a = \lambda y^a, \qquad \tilde{\eta}_{ab} = \eta_{ab}$$

It is possible to show that the ϕ_ρ of (14) transforms according to

$$\tilde{\phi}_\rho = \phi_\rho + \frac{\partial(\ln \psi)}{\partial x^\rho} \tag{16}$$

The internal connection Γ_ρ has the usual value used in general relativity, determined from the metrical conditions on the vierbeins: $e^a_{\mu;\nu} = 0$. Thus both Ω and Γ are independent of the quantities y^a. Consequently these connections are invariant under the transformation (13). From this result it follows that the internal connection F_ρ, under the action of the transformation (15) undergoes an Einstein λ transformation. These results imply that the corresponding curvatures are invariant under the transformations (15), and consequently the gravitational field is presently described by Einstein's Lagrangian density differently from what occcurs in the Weyl theory.

3. THE CURVATURES

For the connections Ω and Γ the curvatures are $R^{\mu}_{.\nu\alpha\beta}$, the Riemann tensor, and $P_{\alpha\beta} = (R^{a}_{.b\alpha\beta})$, similarly to what occurs in general relativity. For the determination of the curvature associated to F_ρ we use the formula

$$y^a_{\|\rho} = y^a_{|\rho} + \tau^a_{.d\rho} y^d = y^a_{|\rho} - \phi_\rho y^a$$

Since this equation may be written in a form similar to the minimal electromagnetic interaction

$$y^a_{\|\rho} = \pi_\rho y^a + \Gamma^a_{.b\rho} y^b, \qquad \pi_\rho = \partial_\rho - \phi_\rho$$

it follows that the corresponding curvature may be given by the expression

$$\begin{aligned}(y^a_{\|\rho})_{|\lambda} - (y^a_{\|\lambda})_{|\rho} &= P^a_{.b\rho\lambda} y^b - F_{\rho\lambda} y^a + \phi_\lambda y^a_{|\rho} - \phi_\rho y^a_{|\lambda} \\ &= Q^a_{.b\rho\lambda} y^b + \phi_\lambda y^a_{|\rho} - \phi_\rho y^a_{|\lambda} \qquad (17)\end{aligned}$$

with

$$F_{\rho\lambda} = \partial_\lambda \phi_\rho - \partial_\rho \phi_\lambda \qquad (18)$$

Owing to the conformal invariance of Ω and Γ under the "conformal transformations" (15) the corresponding curvature tensors are also "conformal invariant." A direct inspection in equations (17) and (18) show that the curvature $Q_{\rho\lambda}$ is conformal invariant. This conclusion also follows from the fact that the connection F_ρ changes under "conformal transformations" according to an Einstein λ transformation.

As it is well known the Ricci tensor and the scalar of curvature of the Riemann tensor may be written in function of the curvature $P_{\alpha\beta}$ as

$$R_{\nu\sigma} = e^\mu_a e^b_\nu P^a_{.b\mu\sigma}, \qquad R = g^{\nu\alpha} e^\mu_a e^b_\nu P^a_{,b\mu\alpha}$$

For the curvature $Q_{\alpha\beta}$, which is given by the expression $Q_{\alpha\beta} = P_{\alpha\beta} - F_{\alpha\beta}.1$, the corresponding Ricci tensor and scalar of curvature are of the form

$$S_{\nu\sigma} = R_{\nu\sigma} - F_{\nu\sigma}, \qquad S = R \qquad (19)$$

4. THE LAGRANGIAN DENSITY

According to the present method, the connection F_ρ differs from the conventional $SO(3,1)$ connection Γ_ρ by the object $-1.\phi_\rho$. From the metrical conditions which have been chosen, the vector field ϕ_ρ is given by equations (14). These equations show that ϕ_ρ differs from the expression of a pure gauge potential owing to the presence of Γ_ρ in the derivative $y^a_{|\rho}$. This shows that this potential is of gravitational origin. From the expression of the curvature $Q_{\rho\lambda}$ we see that there exists a nonnull component of the curvature associated to F_ρ which is given by equations (18) and has the structure of a field strength associated to ϕ_ρ.

Thus, the quantities ϕ_ρ and $F_{\rho\sigma}$ are essentially of geometrical nature. Indeed, if we make $\Gamma_\rho \to 0$ (no gravitation is present, or it may be neglected), we get $P_{\rho\sigma} \to 0$, $F_{\rho\sigma} \to 0$, and consequently both the Riemann tensor and the curvature $Q_{\rho\nu}$ vanish globally. The presence of a component of the curvature with the structure of a field strength suggests as we already referred to above, that the quantities ϕ_ρ should be interpreted as potentials. This conclusion is further substantiated by the fact that ϕ_ρ depends on Γ_ρ, which is a quantity of importance in the dynamics of gravitation according to the previous argument. However, if we try to construct a Lagrangian density based on the curvature $Q_{\rho\sigma}$ such Lagrangian has to be at most quadratic in the curvature. Indeed, it is not difficult to show that

$$\mathrm{Tr}(Q^{\rho\sigma}Q_{\rho\sigma}) = R^{\alpha\nu\rho\sigma}R_{\alpha\nu\rho\sigma} - 4F_{\rho\sigma}F^{\rho\sigma} \tag{20}$$

whereas an expression linear in $Q_{\rho\sigma}$ will contain no information on gravitation,[5] or the scalar of curvature associated to $Q_{\rho\sigma}$ has the conventional Riemannian expression.

Gravitational Lagrangians quadratic in the curvature are known in the literature, but presently we will use a simpler expression which has a formal analogy with the Einstein-Maxwell Lagrangian density:

$$\mathscr{L} = \sqrt{[-g(x)]}^{1/2}R + \alpha\sqrt{[-g(x)]}^{1/2}F_{\rho\sigma}F^{\rho\sigma} \tag{21}$$

This expression differs from the Einstein-Maxwell Lagrangian owing to the fact that here $F_{\rho\sigma}$ is of gravitational origin. Indeed, for the Einstein-Maxwell system it is well known that the vanishing of the components of the Riemann tensor implies in the vanishing of the components of the electromagnetic field strength, but the converse is not true. For the spin-1 field presently considered, this result and the inverse proposition can be obtained purely in geometrical form without appeal to the form of the Lagrangian. Indeed, if gravitation may be neglected, we may choose Cartesian coordinates and set $g_{\mu\nu}(x) = \eta_{\mu\nu}$, $\Omega = 0$ globally. Since then $l^a_\mu = \delta^a_\mu$ it follows that Γ_μ vanishes. From equation (14) we have that $\phi_\mu = \partial(\ln\psi)/\partial x^\mu$ and as a consequence $F_{\mu\nu} = 0$. The converse is also valid: if $F_{\mu\nu} = 0$ we have that $\phi_\mu = \partial x/\partial x^\mu$, and from (14) this implies in $\Gamma_\mu = 0$.

It is well known that this later result implies that the Riemann tensor vanishes.[6] Thus the curvature $Q_{\rho\sigma}$ cannot be dissociated in two distinct terms $P_{\mu\nu}$ and $F_{\mu\nu}$ which would assume independent values.

[5]From the definition of $Q_{\rho\sigma}$ it follows that Tr $Q_{\rho\sigma} = -4F_{\rho\sigma}$, and an obvious scalar formed with this expression by contraction with metric $g^{\rho\sigma}(x)$ vanishes.

[6]In this case the vierbeins determine a field of parallel vectors (absolute parallelism).

5. THE WEAK FIELD APPROXIMATION

Since both $g_{\mu\nu}$ and ϕ_μ are presently of geometrical nature, any first-order approximation must be carried out simultaneously for these two quantities. In this section we want to verify if there will be a first-order contribution of the potentials ϕ_μ to the geometry of the system.

It is natural to take

$$\omega_{ab}(y) = \eta_{ab} + \chi_{ab}(y)$$

with $\chi_{ab}(y)$ of the first order. Since we also have the relation

$$\omega_{ab}(y) = \eta_{ab}\psi(y)$$

with $\psi(y)$ homogeneous of the first degree in the y^a. Then

$$\chi_{ab} = \eta_{ab}(\psi - 1) \tag{22}$$

The choice of $\psi(y)$ is limited by the condition that it cannot depend explicitly of the coordinates X^μ. Thus, it can be, at most, a pure function of the y^a and of the constant tensor η_{ab}. Accordingly, we will take

$$\psi = A\sqrt{(\eta_{cd}y^c y^d)^{1/2}} \tag{23}$$

where A is a constant.

We impose a first-order approximation in the components y^a according to

$$y^a(x) = c^a + \xi^a(x)$$

From (23) we obtain, up to first order terms in the ξ^a, choosing the constant A by the relation $Ac^2 = 1$:

$$\chi(y) = 1 + \frac{c_a\xi^a}{c^2} \tag{24}$$

Then, the tensor χ_{ab} takes the form

$$\chi_{ab} = \eta_{ab}\frac{c.\xi}{c^2} \tag{25}$$

where

$$l^a = c^b l^a_{.b}$$

$$h^a = c^b h^a_{.b} = -c^b(l^a_{.b} + l_{b.}{}^a)$$

For the term giving the derivatives of ln ψ we have

$$\frac{\partial}{\partial y^a}(\ln \psi) = \frac{1}{\psi}\frac{c_b}{c^2}\frac{\partial \xi^b}{\partial y^a} = \left(1 - \frac{c_b\xi^b}{c^2}\right)\frac{c_a}{c^2}$$

Then

$$\phi_\rho = \xi_{,\rho} + l_{,\rho} + \tfrac{1}{2}h_{,\rho}$$

In this expression we are denoting

$$u = \frac{1}{c^2} c_a u^a, \qquad u^a = (\xi^a, l^a, h^a)$$

Therefore, in the first-order approximation the vector ϕ_ρ is the gradient of a scalar function, and accordingly the field strength $F_{\rho\lambda}$ vanishes. Any non-null effect of the field $F_{\rho\lambda}$ begins necessarily in the quadratic order.

6. FINAL COMMENTS

According to the expression $\pi_\rho = \partial_\rho - \phi_\rho$, the potential ϕ_ρ has dimension L^{-1}. Writing $\phi_\rho = kB_\rho$, and taking B_ρ with the dimension $M^{1/2}L^{1/2}T^{-1}$ similarly to the dimension of the electromagnetic potentials, it follows that $\dim k = M^{-1/2}\, TL^{-3/2} = \dim(e/hc)$ where e is electric charge. The remaining quantities like ψ, y, and $g_{\mu\nu}(x, y)$ are taken as dimensionless quantities. It is simple to verify that the expression (12) giving the explicit expression of ϕ_ρ has the dimension L^{-1}, independently of the dimension chosen for ψ and y^a.

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